Solution to Problem 3) The volume of the cylinder, $V(r,h) = \pi r^2 h$, is the function that needs to be maximized. The constraint is $g(r,h) = r^2 + (h/2)^2 = R^2$, which is readily obtained by inspecting the diagram that shows the cylinder encompassed by the sphere. We proceed to optimize the function $V + \lambda g = \pi r^2 h + \lambda[r^2 + (h/2)^2]$ by setting its partial derivatives with respect to r and h equal to zero. We find

$$\partial(V + \lambda g)/\partial r = 2\pi r h + 2\lambda r = 0 \quad \rightarrow \quad h_0 = -\lambda/\pi.$$

$$\partial(V + \lambda g)/\partial h = \pi r^2 + ½\lambda h = 0 \quad \rightarrow \quad r_0^2 = \lambda^2/2\pi^2.$$

Substitution into the constraint equation, namely, $g(r,h) = R^2$, now yields

$$g(r_0, h_0) = \lambda^2/(2\pi^2) + \lambda^2/(4\pi^2) = ¾(\lambda/\pi)^2 = R^2 \quad \rightarrow \quad \lambda_0 = \pm\frac{2\pi}{\sqrt{3}}R.$$

With the value of λ_0 at hand, we substitute in the expressions for r_0 and h_0 to determine the optimum values of the cylinder's radius and height. The positive value of λ_0 yields a negative value for h_0, which is unacceptable. Therefore,

$$h_0 = 2R/\sqrt{3}, \qquad r_0 = \sqrt{2/3}\,R, \qquad V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}.$$

The maximum volume of the cylinder is thus seen to be equal to the volume of the sphere divided by $\sqrt{3}$.